No. I.

A method of finding the Area of a field arithmetically, *by* ELIZUR WRIGHT, A. M.

THE inveſtigation of the area of a field, performed according to the two modes of operation, Geometrical and Arithmetical, has in each its peculiar excellences and defects. The lines and figures, ariſing in a geometrical operation, aſſiſt the imagination, and give an exact image and idea of the form of the field. The operation is at the ſame time eaſy and expeditious. But the arithmetical method is capable of a higher degree of accuracy, and enables us to approach to a greater nearneſs to truth. The minutes of a degree, or the decimal parts of a chain, when geometrically conſidered, approach ſo near to a point, that it often becomes impracticable to lay them down on paper; which occaſions a ſenſible error in the operation, ſo that ſurveyors will frequently differ very conſiderably in computing the area of the ſame field. For theſe reaſons it will be doing an acceptable ſervice to furniſh the ingenious artiſt with an univerſal method of finding the area of a field by numbers, that ſhall exhibit an eaſy and natural order, ſo that the operator may proceed without perplexity or danger of miſtake: and alſo afford a ſhort and concinnous ſolution, performed with the leaſt poſſible number of figures. Having had occaſion to turn my thoughts to the computation of areas, I hit upon a numerical method of find-

ing the area of a field, which I ſhall attempt to communicate, as it occurred to me, in the following manner :

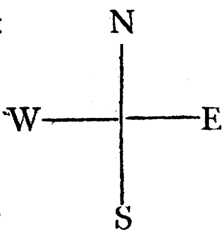
Let it be ſuppoſed that the ſides A B, B C, C D, D A, (Plate I. fig. 6,) of a field are hypothenuſes of right angled triangles, the baſes of which are eaſt and weſt lines, and are either eaſtings, as E B, F C, or weſtings, as G D, H A ; and the perpendiculars north and ſouth lines, being either northings, as A E, B F, or ſouthings as C G, D H according to the bearings of the ſides. The eaſtings, weſtings, northings, and ſouthings are denoted by the initial letters E. W. N. S. in the expreſſion of the courſe. The *calculatrix* is an eaſt and weſt line, which may poſſeſs any aſſignable place E W, ew (Plate I. fig. 7,) *ad libitum* : but to ſhorten the operation it is ſuppoſed to biſect the firſt ſide, with which the operation commences. A perpendicular line C O, drawn from the further extremity of a ſide B C to the calculatrix, is the ultimate *calculatral diſtance* of that ſide ; and a perpendicular S M, drawn from the middle of a ſide B C, is the *mean calculatral diſtance* of that ſide. In the following method the calculatral diſtance is taken double. To the triangle A B N add the trapezium N B C O ; the ſum is the area A B C O ; from which ſubtract the trapezium O C D P, and it leaves the area A B C D P ; and from this ſubtract the triangle P D A, and the remainder is the area of the field A B C D. The areas of theſe triangles and trapezia, and from thence the area of the field may be obtained by the following

RULE.

Make a divifion in the field book of two columns; in the firft of which note down the courfes and fides of the field, as they occur. Find the bafes and perpendiculars of the feveral fides, which may be expeditioufly done by the help of a table. Place the bafes in the firft column underneath the fides, and the perpendiculars in the fecond column, prefixing the *fign* — to thofe that are *fouthings*. Let the *calculatrix* be confidered as *bifecting* the *firft fide*; then the *mean calculatral diftance* of the *firft fide* is *equal to O*; and its *perpendicular* is equal to its *ultimate calculatral diftance*. When they are *both northings*, or *both fouthings*, take the *fum* of the *ultimate calculatral diftance* of the firft fide, and the *perpendicular* of the fecond; and it will be the *mean calculatral diftance* of the fecond fide. Again, take the *fum* of the aforefaid *perpendicular* and *mean calculatral diftance*; and it will be the *ultimate calculatral diftance* of the fecond fide. But when they are, *one a northing* and the *other a fouthing*, take their *difference* inftead of fum. Here let it be obferved, that the mean and ultimate calculatral diftance is of the fame name with the number added to, or fubtracted from; and when it is a fouthing, it muft be defignated by prefixing the fign — to it. In like manner proceed with each of the remaining fides. Multiply the *mean calculatral* diftance of each fide by its *bafe*. Then bring the feveral products, whofe factors are northings and eaftings, or fouthings and weftings, into one fum: likewife bring the products, whofe

factors

factors are northings and westings, or southings and eastings, into another sum. To assist the memory let the annexed diagram represent the cardinal points; and let it be conceived, that the products, to be brought into one sum, have their factors in the order N E, S W; and that those, to be brought into the other sum, have their factors in the contrary order N W, S E. Lastly take the *difference* of these *sums*; *half* of which will be the required *area* of the field.



EXAMPLE I.

C. S. B.	P.
S. 69° 43′ W. 21.12 19 81	—7.32 0.00 —7.32
N. 55° E. 18 14 74	10.32 3.00 13.32
S. 42° W. 6 4 01	—4.46 8.86 4.40
N. 64° E. 12 10 78	5.27 9.67 14.94
S. 24° W. 4 17 1 70	—3.81 11.13 7.32
25 52 25 52 N E. S W. 44.2200 104.2426	N W. S E. 35.5286 18.9210
148.4626 54.4496	54.4496
94.0130 47.0065	

Suppose the survey of a field to be as follows: From the first boundary S. 69° 43′ W. 21.12 chains: thence N. 55° E. 18 chains: thence S. 42° W. 6 chains: thence N. 64° E. 12 chains: thence S. 24° W. to the first boundary. The required area is 47.0065 chains, or 4.70065 acres.

When the sum of the eastings is equal to the sum of the westings, and the sum of the northings is equal to the sum of the southings; which is known by the ultimate calculatral distance of the last side being equal to the perpendicular of the first side, but of a different name, the survey is rightly taken; otherwise there is an error either in the courses, or sides.

In

In order to demonſtrate this numerical method of finding the area, I ſhall lay down the following Lemmas.

Lem. 1ſt. In the trapezium R G H Q (Plate I. fig. 8,) right angled at R, and Q; if to twice G R be added O H, and this ſum be multiplied by G O, the product will be equal to twice the area of the trapezium. For 2 G R×G O=2 R G O Q, and O H×G O=2 G H O. Therefore $\overline{2\ GR+OH}\times GO=$ 2 R G H Q. Q. E. D.

Lem. 2d. In the foregoing trapezium if from twice H Q be ſubtracted H O, and the remainder be multiplied by G O, the product will be equal to twice the area of the trapezium. For 2 H Q×G O=2 R A H Q, and H O×G O=2 G A H O. Therefore $\overline{2\ HQ-HO}\times GO=2$ R G H Q. Q. E. D.

Lem. 3d. In figure 9th, plate I. if the triangle C F L be made equal to the right angled triangle B O K, then M L multiplied by K M will be equal to twice the area of the trapezium O F L M. For M L×K M=O F L M+O F A K= 2 O F L M. Q. E. D.

The diagram A B C D E (Plate I. fig. 10,) is a geometrical conſtruction of the foregoing example. K Q is the calculatrix. K M, D T are eaſtings, and I K, N D, P A are weſtings. B K+M C, T E are northings, and A I+K B, C N, E P are ſouthings. C M, D R, E Q, A I are the calculatral diſtances of the ſides B C, C D, D E, E A. The operation according to the conſtruction is as follows.

N E,

C. S. B.	P.
S. 69° 43′ W. A.B. IK	—AI—KB O —AI—KB
N. 55 E. BC KM	BK+MC LM 2 CM
S. 42 W. CD ND	—CN 2 CM—CN 2 DR
N. 64 E. DE DT	TE 2 DR+TE 2 EQ
S. 24 W. EA PA	—EP 2 EQ—EP AI+KB

NE, SW.

2 OFLM ($=$ML$\times$KM by Lem. 3.)

2 DEQR ($=\overline{2\ DR+TE}\times$DT by Lem. 1.)

NW, SE.

2 CDRM ($=\overline{2\ CM-CN}\times$ND by Lem. 2.)

2 EAIQ ($=\overline{2\ EQ-EP}\times$PA by Lem. 2.)

OFLM+DEQR—CDRM—EAIQ=ABCDE.

DEMONSTRATION.

OFLM+FCL=OCM.

OCM—CDRM=OCDR.

OCDR+DEQR=OCDEQ.

OCDEQ—EAIQ=OCDEAI.

OCDEAI=OCDEABK.

OCDEABK—BOK=ABCDE.

Hence by fubftitution

OFLM—CDRM+DEQR—EAIQ=ABCDE.

Q. E. D.

EXAMPLE

EXAMPLE 2.

C. S. B.	P.	N E. S W.	N W. S E.
N. 40E. 10 6.43	7.66 0.00 7.66	93.7388 5.9826 15.7384	32.8680 .7398 17.5200
N. 56W. 4 3.32	2.24 9.90 12.14	115.4598 51.1278	51.1278
S. 80E. 9 8.86	—1.56 10.58 9.02	64.3320 Area is 32.1660	
S. 25E. 8 3.38	—7.25 1.77 —5.48		
N. 36W. 7 4.11	5.66 .18 5.84		
W. 3 3.00	0.00 5.84 5.84		
S. 1 0	—1.00 4.84 3.84		
S. 55° 9′ W. 10 04 8.24	—5.75 —1.91 —7.66		
18.67 18.67			

No. II.

A method of finding the area of a Field by east and west areas.

Aſſume 10, or 100, or 1000 according to the dimenſions of the field for the mean calculatral diſtance of the firſt ſide. Add the perpendicular of the firſt ſide, if a northing to its mean calculatral diſtance; and the ſum is its ultimate calculatral diſtance. But if the perpendicular is a ſouthing, ſubtract inſtead of adding. The perpendicular of each of the remaining ſides

sides must be *twice added* if a *northing*, but if a *southing*, *twice subtracted* from the ultimate calculatral distance of the preceding side ; the *first* obtains the *mean calculatral distance*, and the *second* the *ultimate calculatral distance* of that side. Multiply the *mean calculatral distance* of each side by its *base*. Then take the *difference* of the *East* and *West* products or areas, *half* of which will be the required *area* of the field.

EXAMPLE.

C. S. B.	P.	E. Areas.	W. Areas.
S. 69 43 W. 21 12	—7.32	191.6200	198.1000
	10.00	212.0426	75.6286
19 81	2.68		35.9210
		403.6626	
N. 55E. 18	10.32	309.6496	309.6496
	13.00		
14.74	23.32	94.0130	
		Area is	
S. 42 W. 6	—4.46	47.0065	
	18.86		
4.01	14.40		
N. 64E. 12	5.27		
	19.67		
10.78	24.94		
S. 24W. 4.17	—3.81		
	21.13		
1.70	17.32		
25.52 25.52	10.00		

No. III.

A general solution of the problem to find the area of an irregular polygon, having the sides and angles given.

Let the several sides of the polygon A B, B C, C D, D E, E A (Plate I. fig. 11,) be considered as hypothenuses of right angled triangles, of which the perpendiculars B F, C G, D H, E I are parallel to the prime side A B, or the side with which the operation

ation begins. Alſo let the given angles A B C, B C D, C D E, &c. be exchanged for the angles F B C, G C D, &c. Theſe angles of *commutation* are obtained by methods hereafter deſcribed. In order to perform the operation, the ſeveral poſitions of the baſes, and perpendiculars, muſt be diſcovered, and deſignated. The perpendiculars on one ſide of the baſes, as F B, G C are denoted by A; and the perpendiculars on the oppoſite ſide, as H D, I E by B, prefixed to the angle of commutation. Likewiſe the baſes on one ſide of the perpendiculars, as F C, G D are denoted by A; and the baſes on the oppoſite ſide, as H E, I A by B, ſuffixed to the angle of commutation. It is evident, that when a ſide happens to be at right angles with the prime ſide, the hypothenuſe and its baſe become equal; and when a ſide is parallel with the prime ſide, the hypothenuſe and its perpendicular become equal: and in the former caſe the perpendicular, and in the latter the baſe and angle of commutation evaniſh, and become equal to O. Yet for the ſake of diſcovering the poſitions of the ſequent baſes, and perpendiculars, they muſt be retained, and brought into the calculation. For this end they may be conſidered as indefinitely ſmall quantities, or quantities leſs than any aſſignable one; and will therefore be expreſſed by O. Place either A, or B before, and either A, or B after the evaneſcent angle of commutation belonging to the prime ſide. Then the an-

gles of commutation may be found by the following rule, which contains two cafes.

CASE I. When the letters affixed to the angle of commutation laft found, are the *fame*.

Take the *fum* of the angle of *commutation* laft found and the *given* angle, if the given angle be *inward:* but if it be *outward*, take their *difference*.

CASE II. When the letters affixed to the angle of commutation laft found, are *different*.

Take the *difference* of the angle of *commutation* laft found and the *given* angle, if the given angle be *inward:* but if it be *outward*, take their *fum;* and the refult is denominated the *factum*. In both cafes when the factum is lefs than 90, it is the angle of commutation fought: but when it exceeds 90, fubtract it from 180; and when it exceeds 180, fubtract 180 from it, and the remainder will be the required angle of commutation.

Nextly the pofitions of the bafes and perpendiculars may be eafily difcovered from the foregoing operation by infpection thus:

CASE I. When the factum is *lefs* than 90.

PROP. 1. If the angle of *commutation* be *fubtracted* from the *given* angle, the letters prefixed to the two angles of commutation laft found, will be *unlike*, and the letters fuffixed will be *like*.

PROP. 2. If the angle of *commutation* be *not ſubtracted* from the *given* angle, the letters prefixed will be *unlike*, and the letters ſuffixed *unlike*.

CASE II. When the factum is *greater* than 90.

PROP. 1. If the *ſum* of the angle of *commutation* and *given* angle be *ſubtracted* from 180, the letters prefixed will be *like*, and the letters ſuffixed *unlike*.

PROP. 2. If the aforeſaid *ſum* be *not ſubtracted* from 180, the letters prefixed will be *like*, and the letters ſuffixed *like*.

When the factum is 90, it may be conſidered as belonging either to Caſe I, or Caſe II; and when the factum is 180, it may be conſidered as belonging either to Caſe II. Prop. 1, or Caſe II. Prop. 2, ad libitum; each ſuppoſition leading to a true diſcovery of the poſitions of the ſubſequent baſes and perpendiculars.

The *calculatrix* is a line K L (Plate I. fig. 12.) biſecting the ſecond ſide BC, and at right angles with the prime ſide A B.

A perpendicular D L drawn from the farther extremity of a ſide C D to the calculatrix, is the ultimate *calculatral diſtance* of that ſide.

A perpendicular M S, drawn from the middle of a ſide C D, is the *mean calculatral diſtance* of that ſide.

In the following method the calculatral diſtance is taken double.

double. Find the bafes and perpendiculars of the feveral fides ; placing the bafes in the fecond column underneath the fides, and the perpendiculars in the third column. Prefix the *negative fign* — to thofe bafes and perpendiculars, whofe pofitions are defignated by *B*. Paffing over the prime fide, the *mean calculatral diftance* of the *fecond* fide is *equal to O ;* and its *perpendicular* is equal to its *ultimate calculatral diftance*. Take the *fum* of the *ultimate calculatral diftance* of the fecond fide, and the *perpendicular* of the third according to the rules of Algebra, and it will be the *mean calculatral diftance* of the third fide. Again take the *fum* of the aforefaid *perpendicular* and *mean calculatral diftance*, and it will be the *ultimate calculatral diftance* of the third fide. In like manner proceed with each of the remaining fides. Multiply the *mean calculatral diftance* of each fide by its *bafe :* then *half* the Algebraic *fum* of thefe products will be the required *area* of the polygon. It may here be obferved, that when the work is done right, the angle of commutation for the laft fide is equal to the given angle at the beginning of the operation : alfo the fum of the affirmative bafes is equal to the fum of the negative bafes ; and the fum of the affirmative perpendiculars is equal to the fum of the negative, which is the cafe, when twice the perpendicular of the prime fide being added to the ultimate calculatral diftance of the laft fide, the refult is equal to the perpendicular of the fecond fide, but of a contrary value.

EXAMPLE

EXAMPLE I.

A.	S. B.	P.
14°. 43′ A. o A.	18 o	18
13° B. 13 B.	6 —1.35	—5.85 0.00 —5.85
22°∠ A. 9 B.	12 —1.88	11.85 6.00 17.85
40° B. 31 B.	4.17 —2.14	—3.57 14.28 10.71
134°. 17′ 165. 17 B. 14. 43 A.	21.12 5.37	—20.43 —9.72 —30.15
	5.37 5.37	36.00 5.85

—31.2800
30.5592
52.1964

94.0356

Area is
47.0178.

EXAMPLE 2.

A.	S. B.	P.
15° 4′ B. o A.	10 o	—10
84°∠ A. 84 B.	4 —3.98	.41 .00 .41
24° B. 60 A.	9 7.79	—4.50 —4.09 —8.59
125° A. 65 A.	8 7.25	3.38 —5.21 —1.83
11° B. 76 B.	7 —6.79	—1.69 —3.52 —5.21
126°∠ A. 50 B.	3 —2.30	1.93 —3.28 —1.35
90°∠ 140 A. 40 A.	1 64	.77 —.58 .19
124°. 56′ 164. 56 A. 15. 4 B.	10.04 —2.61	9.70 9.89 19.59
	15.68 15.68	—20.00 —.41

+23.9008
7.5440

31.4448

—31.8611
37.7725
0.3712
25.8129

95.8177
31.4448

64.3729

Area is
32.1864

EXAMPLE I.

In the Polygon A B C D E there is given the angles E A B $=14^\circ\ 43'$, A B C$=13^\circ$, B C D$=22^\circ\angle$, C D E$=40^\circ$, D E A$=134^\circ\ 17'$; and the fides A B$=18$, B C$=6$, C D$=12$, D E$=4.17$, E A$=21.12$; required the area.

EXAMPLE 2.

In the polygon A B C D E F G H there is given the angles H A B$=15^\circ\ 4'$, A B C$=84^\circ\angle$, B C D$=24^\circ$, C D E$=125^\circ$, D E F $=11^\circ$,

$=11^\circ$, EFG$=126^\circ\angle$, FGH$=90^\circ\angle$, GHA$=124^\circ\ 56'$; and the ſides AB$=10$, BC$=4$, CD$=9$, DE$=8$, EF$=7$, FG$=3$, GH$=1$, HA$=10.04$; required the area.

The magnetical needle is found to vary ſeveral minutes from a true traverſe. This is occaſioned by friction. Where a great accuracy therefore is required in taking the ſurvey of a field, this general rule for finding the polygonal area may be very advantageouſly applied. For this end take the menſuration of the including angles of the field, independent of the needle, which may be done with great exactneſs with a good inſtrument. Then having meaſured the ſides with care and exactneſs, we have the angles and ſides of an irregular polygon given to find the area. If the bearings of the ſeveral ſides are deſired, they may in the following manner be made to ariſe from the calculation itſelf. Interpolate the polygon, at the prime angle, with an indefinitely ſhort ſide, being a meridian line, had by celeſtial obſervations, or the magnetical needle. Take the angles, which this ſide makes with the adjoining ſides of the polygon, and note them down with the reſt. This infiniteſſimal ſide is to be conſidered as the prime ſide, and may be expreſſed by O. Having obtained the angle of commutation for the ſecond ſide, inſtead of the deſignating letters A, B, introduce the initial letters of the cardinal points proper to that ſide; diſtinguiſhing thoſe that ſtand in the room of B by a daſh over the top. The angles of commutation will now become the points of compaſs, and expreſs the true bearing of each ſide.

EXAMPLE

EXAMPLE 1.

A. C.	S. B.	P.
$69^{\circ}.\ 43'$ A. o A.	o o	o
125° 125 A. 55 B. N. 55 $\overline{\text{C}}$.	18 14.74	10.32 0.00 10.32
-13° $\overline{\text{S}}$. 42 W.	6 4.01	—4.46 5.86 1.40
$22^{\circ}\angle$ N. 64 $\overline{\text{E}}$.	12 10.78	5.27 6.67 11.94
40° $\overline{\text{S}}$. 24 W.	4.17 1.70	—3.81 8.13 4.32
$134^{\circ}.\ 17'$ 110. 17 $\overline{\text{S}}$. 69.43 W.	21.12 19.81	—7.32 —3.00 —10.32
N W. S E.	N E. S W.	
23.4986 13.8210 37.3196	71.9026 59.4300 131.3326 37.3196 94.0130 Area is 47.0065	

EXAMPLE 2.

A. C.	S. B.	P.
$55^{\circ}.\ 9$ B. o B.	o o	o
140° B. 40 A. $\overline{\text{N}}$. 40 E.	10 6.43	7.66 100.00 107.66
$84^{\circ}\angle$ 124 $\overline{\text{N}}$. 56 $\overline{\text{W}}$.	4 3.32	2.24 109.90 112.14
24° S. 80 E.	9 8.86	—1.56 110.58 109.02
125° 205 S. 25 E.	8 3.38	—7.25 101.77 94.52
11° $\overline{\text{N}}$. 36 $\overline{\text{W}}$.	7 4.11	5.66 100.18 105.84
$126^{\circ}\angle$ S. 90 $\overline{\text{W}}$.	3 3.00	—0.00 105.84 105.84
90° 180 S. o E.	1 0.00	—1.00 104.84 103.84
$124^{\circ}.51'$ 124 .51 S. 55.9 $\overline{\text{W}}$.	10.04 8.24	—5.75 98.09 92.34
	18.67 18.67	7.66 100.00

W. Area.

364.8680
411.7398
317.5200
808.2616

1902.3894

E. Area.

643.0000
979.7388
343.9826

1966.7214
1902.3894

64.3320

Area is

32.1660.

Canaan, (Connecticut) March 20, 1792.